Long-time fluctuations in a dynamical model of stock market indices

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Financial time series typically exhibit strong fluctuations that cannot be described by a Gaussian distribution. Recent empirical studies of stock market indices examined whether the distribution $P(r)$ of returns $r(\tau)$ after some time τ can be described by a (truncated) Lévy-stable distribution $L_\alpha(r)$ with some index $0<\alpha\leqslant 2$. While the Lévy distribution cannot be expressed in a closed form, one can identify its parameters by testing the dependence of the central peak height on τ as well as the power-law decay of the tails. In an earlier study [R. N. Mantegna and H. E. Stanley, Nature (London) **376**, 46 (1995)] it was found that the behavior of the central peak of $P(r)$ for the Standard & Poor 500 index is consistent with the Lévy distribution with $\alpha=1.4$. In a more recent study [P. Gopikrishnan *et al.*, Phys. Rev. E **60**, 5305 (1999)] it was found that the tails of $P(r)$ exhibit a power-law decay, with an exponent $\alpha\cong 3$, thus deviating from the Lévy distribution. In this paper we study the distribution of returns in a generic model that describes the dynamics of stock market indices. For the distributions $P(r)$ generated by this model, we observe that the scaling of the central peak is consistent with a Lévy distribution while the tails exhibit a power-law distribution with an exponent $\alpha>2$, namely, beyond the range of Lévy-stable distributions. Our results are in agreement with both empirical studies and reconcile the apparent disagreement between their results.

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I. INTRODUCTION

Financial time series are generated by complex dynamical processes that exhibit strong correlations between many degrees of freedom. The efforts to understand the dynamics of economic systems have involved empirical studies in which the temporal fluctuations of the prices of individual companies as well as of stock market indices such as the Standard & Poor 500 (S&P500) were examined [1–8]. These fluctuations can be characterized by the distribution of stock market returns as well as the volatility, which quantifies the magnitude of the market fluctuations.

Consider a stock market index $\overline{W}(t)$. Its value is proportional to the average of the market values W_i, $i=1,\ldots,N$ (given by the stock price of each firm times the number of its outstanding shares) of the N stocks that are included in this index. The fluctuations of $\overline{W}$ can be expressed in terms of the returns after a period of time τ (say, in minutes), given by

$$r(\tau)=\ln \overline{W}(t+\tau)-\ln \overline{W}(t). \tag{1}$$

For any value of τ one can examine the distribution $P(r)$ of the returns $r(\tau)$. The number of independent data points available in the distribution is given by T/τ, where T is the time period covered in the available data set. It was observed long ago that such distributions exhibit slowly decaying tails, unlike the Gaussian or exponential distributions. Moreover, the shape of the distribution was found to exhibit a self-similar form for different choices of τ. It was proposed by Mandelbrot [9] that $P(r)$ may be expressed by a Lévy-stable distribution $L_\alpha(r)$, where $0<\alpha\leqslant 2$ [10,11]. Mathematically, the Lévy distribution $L_\alpha(r)$ is the limit $n\to\infty$ of the distribution of the sum of n independent stochastic variables taken from a power-law distribution of the form $p(r)\sim r^{-1-\alpha}$ when $0<\alpha\leqslant 2$ (which clearly exhibits an infinite variance). This is unlike the case of a distribution with a finite variance, which leads to a Gaussian distribution of the sum, according to the central limit theorem. The Lévy distribution thus exhibits an infinite variance. However, in practical applications its tail is truncated due to an upper cutoff in the power-law distribution that generated it [12]. Although the Lévy distribution cannot be expressed in a closed form [13], it has two scaling properties that can be used in order to examine whether a distribution $P(r)$ obtained from empirical data or numerical simulations is a (truncated) Lévy distribution and to calculate its index $0<\alpha\leqslant 2$. The first property involves the dependence of the central peak height on the time τ, and takes the form [13]

$$L_\alpha(r=0)\sim \tau^{-1/\alpha}. \tag{2}$$

Thus, if the distribution of returns $P(r)$ is a (truncated) Lévy distribution, the value of α can be obtained from the slope of the graph of $P(r=0)$ vs τ on a log-log scale. The second property involves the power-law decay of the tails of the distribution which follows [13]

$$L_\alpha(r)\sim r^{-1-\alpha}. \tag{3}$$

Therefore, if the distribution $P(r)$ is a (truncated) Lévy distribution, the value of α can also be obtained from the slope of the tail of $P(r)$ vs r on a log-log scale. Obviously, a Lévy distribution should satisfy the scaling relations for both the central peak and the tail, with the same exponent α.

The distribution $P(r)$ of the returns $r(\tau)$ for the S&P500 stock market index was recently studied for a range of τ values, using the data for the six-year period of 1984–89 [1]. The scaling of the central peak height vs τ was examined within the range of $1\leqslant\tau\leqslant 1000$ min yielding a straight line

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in the log-log scale over three orders of magnitude, with a slope that corresponds to $\alpha=1.4$. It was thus concluded that $P(r)$ takes the form of a truncated Lévy distribution $L_\alpha(r)$ with the index $\alpha=1.4$. More recently the data set was extended to cover a 13-year period (1984–96) and was examined using a scaling analysis of the tail of the distribution $P(r)$ of the returns $r(\tau)$ for τ in the range between 1 min and 4 days. [2]. It was found that the tail of $P(r)$ vs r on a log-log scale exhibits a straight line domain, indicating a power-law dependence given by Eq. (3). However, the slope was found to be consistent with α in the range $2.5<\alpha<3.5$, where the precise value depends on details such as the value of τ and the fitting procedure. Clearly, these values of α are well outside the Lévy-stable range of $0<\alpha\leqslant 2$. Therefore, not only is the distribution $P(r)$ not a Lévy distribution with $\alpha=1.4$—it is not a Lévy distribution at all. This result appears to be in disagreement with the conclusions of Ref. [1]. We thus observe that while the central peak maintains its Lévy features the tails show a non-Lévy behavior. In order to understand these puzzling results one needs to combine theoretical studies, suitable models, and simulations of stock market dynamics, complementary to the empirical analysis.

In this paper we study the distribution of the returns $P(r)$ in a dynamical model that describes the time evolution of stock market indices [14–17]. The model consists of dynamic variables w_i, $i=1,\ldots,N$ that represent the capitalization (total market values) of N firms. The dynamics represents the increase (or decrease) by a random factor $\lambda(t)$ [taken from a predefined distribution $\Pi(\lambda)$] of the value w_i of the firm i between times t and $t+1$. The dynamical rules also enforce a lower bound on the w_i's, which is a certain fraction $0\leqslant c<1$ of the momentary average of the w_i's. This lower bound may represent the minimal requirements for a company stock to be publicly traded. It turns out that after some equilibration time the w_i's exhibit a power-law distribution of the form $p(w)\sim w^{-1-\alpha}$ [17]. For any given value of N, the exponent $\alpha>0$ is a monotonically increasing function of c. Since $r(\tau)$ can be considered as a sum of τ random variables taken from a power-law distribution $p(w)$, one may expect it to converge to the Lévy distribution $L_\alpha(r)$ with the same exponent α. Since the power-law distribution is truncated from above, the tail of the resulting Lévy distribution is also expected to be truncated [12]. Clearly, the dynamics is much more complicated. One reason for this is that the τ random variables are not completely independent—they are taken from a finite set of N values of the w_i's. Moreover, these values slowly change during the calculation of $r(\tau)$, because at each time step one of the w_i's is updated.

To analyze the distribution of returns $P(r)$ we first tune the parameter c (for the given value of N) to adjust the power-law distribution to the economically relevant case of $\alpha=1.4$ [1,18]. We then examine the distribution of returns $P(r)$ for a range of time intervals τ and test the scaling behavior of the central peak as well as of the tails. It is found that the scaling of the central peak is consistent with a truncated Lévy distribution with $\alpha=1.4$ for a broad range of $1\leqslant\tau\leqslant 1000$. For small values of τ, up to about $\tau=50$ (for $N=1000$) the power-law decay of the tail of $P(r)$ is also consistent with a truncated Lévy distribution with the same value of α. However, for larger values of τ the tail of $P(r)$ exhibits a power-law decay consistent with $\alpha>2$, and thus deviates from the Lévy distribution. These results are in agreement with the empirical analysis of the central peak presented in Ref. [1] as well as with the more recent analysis of the tails presented in Ref. [2]. They thus reconcile the apparent disagreement between these two empirical studies.

The paper is organized as follows. In Sec. II we present the model. Simulations and results are reported in Sec. III, followed by a summary in Sec. IV.

II. THE MODEL

The model [14,15,17] describes the evolution in discrete time of N dynamic variables $w_i(t)$, $i=1,\ldots,N$. At each time step t, an integer i is chosen randomly in the range $1\leqslant i\leqslant N$, which is the index of the dynamic variable w_i to be updated at that time step. A random multiplicative factor $\lambda(t)$ is then drawn from a given distribution $\Pi(\lambda)$, which is independent of i and t and satisfies $\int_\lambda \Pi(\lambda)d\lambda=1$. This can be, for example, a uniform distribution in the range $\lambda_{min}\leqslant\lambda\leqslant\lambda_{max}$, where λ_{min} and λ_{max} are predefined limits. The system is then updated according to the following stochastic time evolution equation:

$$w_i(t+1)=\lambda(t)w_i(t),$$

$$w_j(t+1)=w_j(t), \quad j=1,\ldots,N;\ j\neq i. \tag{4}$$

This is an asynchronous update mechanism. The average value of the system components at time t is given by

$$\overline{w}(t)=\frac{1}{N}\sum_{i=1}^{N} w_i(t). \tag{5}$$

The term on the right-hand side of Eq. (4) describes the effect of autocatalysis at the individual level. In addition to the update rule of Eq. (4), the value of the updated variable $w_i(t+1)$ is constrained to be larger than or equal to some lower bound which is proportional to the momentary average value of the w_i's according to

$$w_i(t+1)\geqslant c\overline{w}(t) \tag{6}$$

where $0\leqslant c<1$ is a constant factor. This constraint is imposed immediately after step (4) by setting

$$w_i(t+1)\rightarrow\max\{w_i(t+1),c\overline{w}(t)\}, \tag{7}$$

where $\overline{w}(t)$, evaluated just before the application of Eq. (4), is used. This constraint describes the effect of autocatalysis at the community level. Numerical simulations of the stochastic multiplicative process described by Eqs. (4) and (7) show that the w_i's follow a power-law distribution of the form

$$p(w)=Kw^{-1-\alpha} \tag{8}$$

for a wide range of lower bounds c, where K is a normalization factor [17]. It was found that the exponent α depends on the parameters c and N and is insensitive to the shape of the probability distribution $\Pi(\lambda)$. For simplicity, we use λ uniformly distributed in the range $0.9 \leqslant \lambda \leqslant 1.1$.

III. SIMULATIONS AND RESULTS

In the simulations below the number of dynamical variables is $N=1000$ and the lower cutoff is chosen as $c=0.3$, the value that provides the economically relevant distribution characterized by $\alpha=1.4$ [1,18]. Under these conditions $p(w)$ exhibits a power-law distribution within three decades, between $w_{min}=0.0003$ and $w_{max}=0.3$. The data for this distribution were obtained from a large number of simulations collecting data at different times within each simulation after some equilibration time. To remove the possible effect of inflation, the values of the w_i's fed into the distribution $p(w)$ were normalized such that at any time t the sum $\Sigma_i w_i(t)=1$, namely, $\overline{w}(t)=1/N$ [17]. In the analysis of the returns, there is no need for such a normalization adjustment, due to the fact that the returns quantify changes relative to the current value of $\overline{w}$, i.e., they are normalized by definition.

Consider the time evolution of the average $\overline{w}(t)$. At each time step, when Eq. (4) is applied, neglecting the effect of the lower cutoff we obtain

$$\overline{w}(t+1)=\overline{w}(t)+\frac{1}{N}[\lambda(t)-1]w_i(t). \tag{9}$$

This can be considered as a generalized random walk with step sizes distributed according to Eq. (8). Therefore, the returns after τ time steps, given by

$$r(\tau)=\ln \overline{w}(t+\tau)-\ln \overline{w}(t), \tag{10}$$

are expected to follow a a truncated Lévy distribution $L_\alpha(r)$ [12]. Note that for small time intervals, the returns given by Eq. (10) coincide with the relative change given by

$$\tilde{r}(\tau)=\frac{\overline{w}(t+\tau)-\overline{w}(t)}{\overline{w}(t)}. \tag{11}$$

However, for large τ these two expressions provide significantly different results.

In Fig. 1 we show the rescaled distribution $\tau^{1/\alpha}P(r/\tau^{1/\alpha})$ of the returns $r(\tau)$ for $\tau=1$, 50, 200, and 1000. Near the central peak the four rescaled graphs collapse into a similar shape. The graphs for $\tau=1$ and 50 maintain a similar rescaled form in the tails also while for larger values of τ the tails go down more sharply.

The value of α that characterizes the distribution can be obtained from the scaling of the central peak height as a function of τ, according to Eq. (2). In Fig. 2 we show the height of the peak $P(r=0)$ as a function of τ on a log-log scale. It is found that the slope of the fit is -0.71, which

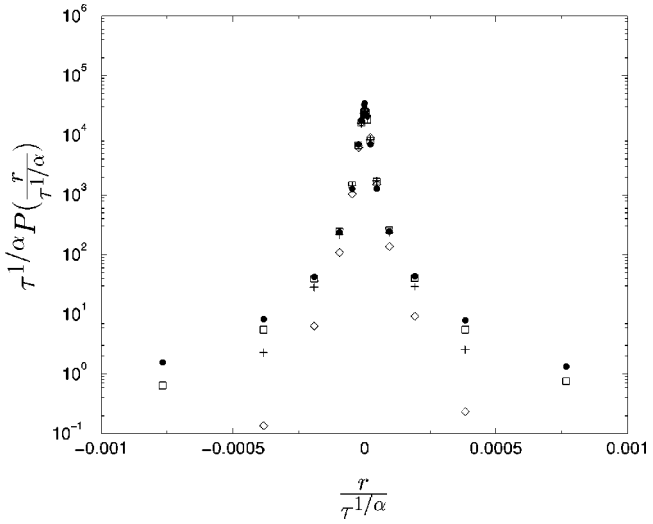

FIG. 1. The rescaled distribution of the returns $\tau^{1/\alpha}P(r(\tau)/\tau^{1/\alpha})$ for $\tau=1$, 50, 200, and 1000. In the vicinity of the central peak we observe a collapse of all four graphs into a similar shape. The tails for the two smaller values of τ follow the Lévy-stable distribution with $\alpha=1.4$. The tails for the two larger values of τ fall off more sharply and exhibit significant deviations from the Lévy-stable shape.

following the scaling relation of Eq. (2) means that the index of the Lévy distribution is $\alpha=-1/(-0.71)=1.4$.

To characterize the nature of the distribution $P(r)$ we also examine the scaling behavior of the tails. For the Lévy distribution the tail is expected to follow a power-law behavior given by Eq. (3). In Fig. 3 we present the tail of the distribution $P(r)$, on a log-log scale for $\tau=1$. It is found that the slope is $-(1+\alpha)=-2.4$ which corresponds to a Lévy distribution with $\alpha=1.4$. For larger values of τ, the tails exhibit steeper slopes that exceed the domain of the Lévy distribu-

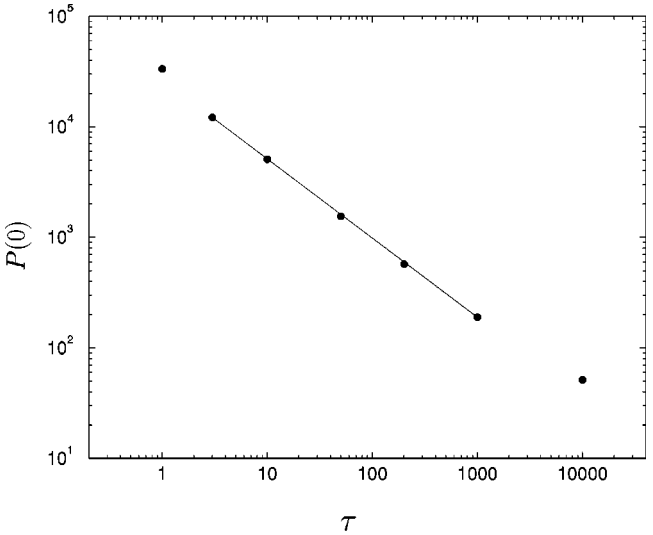

FIG. 2. The height of the central peak $P(r(\tau)=0)$ vs τ on a log-log scale. For a broad range of nearly three orders of magnitude in τ values up to $\tau=1000$, the slope of the straight line is $-1/\alpha=-0.71$, which corresponds to a Lévy distribution with $\alpha=1.4$.

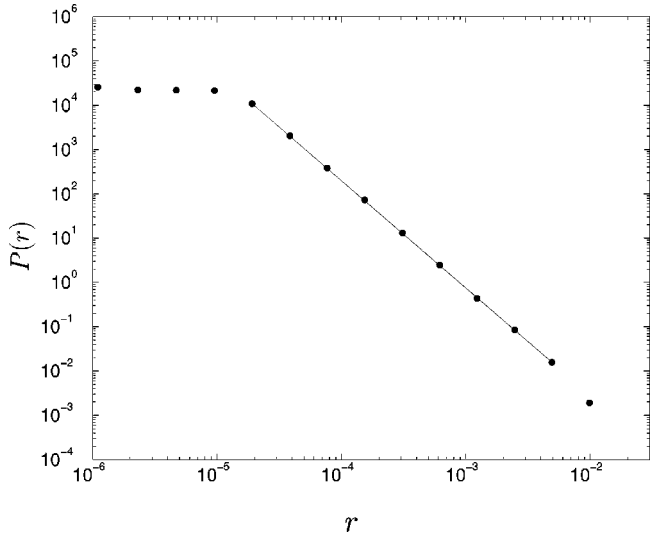

FIG. 3. The distribution $P(r)$ of $r(\tau)$ on a log-log scale, for $\tau=1$. The tail exhibits a range of power-law behavior according to Eq. (3) with $\alpha=1.4$, namely, following a Lévy distribution with the same value of α.

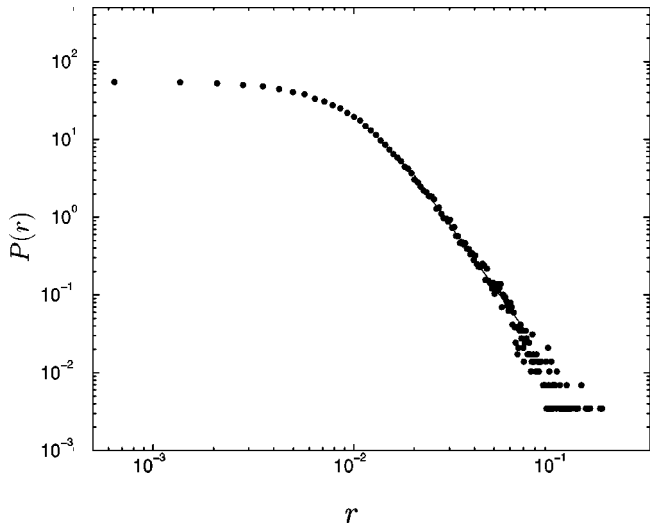

FIG. 4. The distribution $P(r)$ of $r(\tau)$ on a log-log scale, for $\tau=10^4$. The tail exhibits a range of about one order of magnitude with an apparent power-law behavior. The slope in this range is consistent with Eq. (3) with $\alpha=2.5$. This value is not only different from the $\alpha=1.4$ observed for short times, but is outside the range for Lévy-stable distributions. This curve strongly resembles the empirical distributions for the S&P500 presented in Ref. [2].

tion, namely, α becomes larger than 2. As an example, we present in Fig. 4 the distribution $P(r)$ of $r(\tau)$ for $\tau=10^4$ on a log-log scale. We identify a range of about one order of magnitude in which the apparent slope is $-(1+\alpha)=-3.5$, corresponding to $\alpha=2.5$, which is outside the domain of the Lévy distribution. It is thus observed that the tails of the distribution $P(r)$ are much more sensitive to deviations from a Lévy-stable process than is the central peak.

These results are in agreement with the empirical analysis of the central peak presented in Ref. [1] as well as with the analysis of the tails presented in Ref. [2]. They thus reconcile the apparent disagreement between these two empirical studies. To relate the parameters of the model more closely to the empirical studies we note that the typical time required for a single stock market transaction is of the order of 1 min. However, the transactions are done simultaneously in all the stocks included in the index that is analyzed. Therefore, the single transaction-time unit (say, 1 min) roughly corresponds, in the model, to $\tau=N$ time steps. The results of Fig. 4 for $\tau=10^4$ are thus expected to correspond to a time interval of several minutes in the empirical analysis. Indeed, the value of $\alpha=2.5$ obtained in the numerical simulations is only slightly lower than the empirical results obtained for τ in the range between 1 and 512 min.

In the model we observe significant deviations from the Lévy distribution as τ increases toward the order of N. A possible explanation is that at this stage some of the w_i's are already sampled more than once in the sequence of τ time steps required to calculate one instance of $r(\tau)$. This violates the requirement in the construction of a (truncated) Lévy-stable distribution that the τ random variables should be independent. This starts to introduce significant correlations between the different variables that compose $r(\tau)$.

Another correlation effect is intrinsic to the calculation of the returns. Consider the return $r(\tau)$, which is given by

$$r(\tau)=\sum_{t=1}^{\tau}\ln\left[1+[\lambda(t)-1]\frac{w_i(t)}{\overline{w}(t)}\right], \tag{12}$$

where the variable $w_i(t)$ is independently picked at any time t. Note that the return depends on the normalized quantities $w_i'=w_i(t)/\overline{w}(t)$. It is easy to see that the w_i''s are not independent since at any time t they satisfy $\Sigma_i w_i'=N$. This dependence is particularly apparent for the large w_i''s, since if one of them turns out to be extremely large the normalization condition prevents other w_j''s from having values in its vicinity.

IV. SUMMARY

Recent empirical studies of the fluctuations in stock market indices have provided conflicting results. In these studies the distribution $P(r)$ of stock market returns $r(\tau)$ after time τ was examined. The scaling of the central peak of $P(r)$ was found to be consistent with a (truncated) Lévy-stable distribution with index $\alpha=1.4$ [1]. However, the scaling of the tails, for a broad range of τ values between 1 min and a few days, was found to exhibit a power-law behavior with an exponent $\alpha\cong 3$, which is well outside the range of the Lévy distribution [2].

In this paper we have examined the distribution $P(r)$ for a model that describes the dynamics of stock market indices. The model consists of dynamical variables w_i, $i=1,\ldots,N$, that describe the time-dependent market values of N firms, while their average is the corresponding stock market index. It was found that the scaling of the central peak is consistent with a Lévy distribution and its index can be tuned to the economically relevant value of $\alpha=1.4$ by tuning a param-

eter. The tails of the distributions $P(r)$ of the returns $r(\tau)$, for a range of τ values that corresponds to the empirically studied time intervals, were found to exhibit a domain of power-law behavior with $\alpha > 2$, which falls outside the range of the Lévy distribution. These results are fully consistent with the empirical results both for the central peak and for the tails and reconcile the apparent disagreement between them.

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